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— Theoretical Atomic Physics —

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Summer Term

Home Work (2)

Task 6: Electron capture

a) Consider the capture of an electron by the bare ion to form the $2p_{1/2}$ and $2p_{3/2}$ states of hydrogen. Estimate the population ratio between these two states after the capture.
b) Repeat the same for $3d_{3/2}$ and $3d_{5/2}$

Task 7: Isotope shift

Assume that the charge Z in the nucleus of a hydrogen-like atom with mass number A is uniformly distributed. The radius of the nucleus can be estimated as $R \approx 1.3\, A^{1/3}\text{fm}$.
a) Show that the potential is given by

$$V(r) = \begin{cases} \frac{Z}{2R}\left(\frac{r^2}{R^2} - 3\right) & r \le R \\ -\frac{Z}{r} & r > R \end{cases}$$

b) Use perturbation theory to calculate the energy shift of the ground state of a hydrogen-like atom. Evaluate it for hydrogen and the uranium, U^{91+} ion.

Task 8: Commutators (2)

Prove the commutation relations

$$[\boldsymbol{l}, H] = i\hbar c \boldsymbol{\alpha} \times \boldsymbol{p}\,, \tag{1}$$

$$[\boldsymbol{s}, H] = -i\hbar c \boldsymbol{\alpha} \times \boldsymbol{p}\,. \tag{2}$$

Furthermore, conclude that $[\boldsymbol{j}, H] = 0$, where H is the Dirac Hamiltonian

$$H = c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \alpha_0 mc^2 + V(r)\,.$$

Task 9: Commutators (3)

Prove, that the operator $\boldsymbol{l} \cdot \boldsymbol{s}$ commutes with the total angular momentum operator $\boldsymbol{j}$ and its square $\boldsymbol{j}^2$.

Task 10: g-factor of the electron

Show that the Dirac equation describes the interaction between a magnetic field and an electron, and that it yields the correct g-factor. Consider the Dirac Hamiltonian

$$H - mc^2 = \begin{pmatrix} V(r) & c\boldsymbol{\sigma} \cdot \left(\boldsymbol{p} - \frac{1}{c}\boldsymbol{A}\right) \\ c\boldsymbol{\sigma} \cdot \left(\boldsymbol{p} - \frac{1}{c}\boldsymbol{A}\right) & V(r) - 2mc^2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = E \begin{pmatrix} f \\ g \end{pmatrix}$$

a) Show that the non-relativistic limit of the Dirac equation is the Pauli equation.
b) Show that the Pauli equation includes the interaction between the spin and a magnetic field, and that it leads to the g-factor $g = 2$.