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— Computational Quantum Physics —

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Sommersemester

Exercise 4

Task 1: Density operator of a pure state (2 points)

Let ρ denote a density operator. Show that $\mathrm{Tr}(\rho^2) \leq 1$ and that $\mathrm{Tr}(\rho^2) = 1$ if and only if ρ represents a pure state.

Task 2: Pure vs. mixed states (5 points)

Which of the following density operators represent pure and which one mixed states? If the state is pure, determine the state vector. Otherwise, find an ensemble representation.

a) $\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}$ b) $\rho = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ c) $\rho = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$ d) $\rho = \begin{pmatrix} 1/2 & -\mathrm{i}/2 \\ \mathrm{i}/2 & 1/2 \end{pmatrix}$ e) $\rho = \begin{pmatrix} 1/2 & \frac{1-\mathrm{i}}{2\sqrt{2}} \\ \frac{1+\mathrm{i}}{2\sqrt{2}} & 1/2 \end{pmatrix}$

Task 3: Distributed gates (5 points)

Suppose we have the Bell state $|\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$ and the operators

$$A = \sigma_x \otimes \sigma_z \,, \quad B = I \otimes \sigma_x \otimes \sigma_z \otimes I \,, \quad C = I \otimes \mathrm{CNOT} \otimes \sigma_x \quad \text{where} \quad \mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Implement a function that computes the tensor products of quantum states/operators. Use your function to determine the following states:

a) $A\,|\psi\rangle$,

b) $B\,|0{+}10\rangle$ (where $|0{+}10\rangle$ means $|0\rangle \otimes |+\rangle \otimes |1\rangle \otimes |0\rangle$),

c) $B(|\psi\rangle \otimes |\psi\rangle)$,

d) $C(|\psi\rangle \otimes |\psi\rangle)$.

e) The Hadamard operator for a single qubit is defined as $H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$. Use your function to find the Hadamard operator $H^{\otimes n} = \underbrace{H \otimes H \otimes \cdots \otimes H}_{n \text{ times}}$ on n qubits for $n = 2, 3, 4$.

You can compare your result to the formula on page 50 of the lecture notes.