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TPI and HI Jena
— Computational Quantum Physics —

8. June 2018
Summer term

Home Work Topics (Exam)

This course aims for learning and dealing with – more or less – simple computations on many-particle quantum systems, and with some particular focus upon multi-qubit systems. For the final home work (exam) of this course, please, select and work out in good detail, either individually or in groups of up to two students (and, then, to some greater depth), one of the following tasks. We shall agree about your selection latest until June 25th, 2018, on the basis 'first come, first served'. You will have to submit (and present) your results until the last lecture week of this term, however not later than August, 30th, 2018, please.

To solve these tasks, please work out a small (set of) procedures, for instance in Julia, in which you implement all the necessary (sub-) procedures of your code. Prepare a short list of these procedures and for what they can be applied. Moreover, please present this program to all of us, together with some explanations about the underlying physics, and demonstrate and discuss how they work in detail.

In the following, we shall refer to the following pre-defined states, the

- Bell states: $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$;
- diagonal Bell states, i.e., a mixture of the two-qubit Bell states:

$$\rho = p_I \left|\Phi^+\right\rangle\left\langle\Phi^+\right| + p_x \left|\Psi^+\right\rangle\left\langle\Psi^+\right| + p_y \left|\Psi^-\right\rangle\left\langle\Psi^-\right| + p_z \left|\Phi^-\right\rangle\left\langle\Phi^-\right|$$

$$= \frac{1}{2}\begin{pmatrix} p_I + p_z & 0 & 0 & p_I - p_z \\ 0 & p_x + p_y & p_x - p_y & 0 \\ 0 & p_x - p_y & p_x + p_y & 0 \\ p_I - p_z & 0 & 0 & p_I + p_z \end{pmatrix},$$

  where the (classical) weights p_i have to fulfill $\sum_i p_i = 1$;

- n-qubit GHZ states: $|\text{GHZ}\rangle_n = \frac{1}{\sqrt{2}}(|00\ldots0\rangle + |11\ldots1\rangle)$;
- n-qubit W states: $|W\rangle_n = \frac{1}{\sqrt{n}}(|00\ldots01\rangle + |00\ldots10\rangle + \cdots + |10\ldots00\rangle)$;
- n-qubit symmetric Dicke states with e excitations (Mandel and Wolf 1995):

$$|n,e\rangle = \binom{N}{e}^{1/2} \sum_i P_i(|1_1, 1_2, \ldots, 1_e, 0_{e+1}, \ldots, 0_N\rangle),$$

  where $\{P_i\}$ refers to the set of all distinct permutations of the qubits. The state $|n,1\rangle$ is the same as the n-qubit W state.

The references given below can be provided by us on request.

Task (Multi-qubit Stokes parameter):

The Stokes parameter of a given (mixed) state provide a simple means to support the tomographic reconstruction of a given density matrix, i.e., a quantum system in a given state.

a) Evaluate and compare graphically the multi-qubit Stokes parameter of the Bell states $|\Psi^{\pm}\rangle$, the diagonal Bell states with selected weights;

b) Calculate and discuss the same for the 3- and 4-qubit W and GHZ-states;

c) Calculate the Stokes parameter also for the 3-qubit symmetric Dicke states.

Reference: J.B. Altepeter *et al.*, Adv. Atomic Molecular and Optical Physics **52** (2005) 105.

Task (Measurement statistics in projective measurements):

For some quantum system in a given state, the probability for outcome m of a measurement can be calculated by following the postulate about quantum measurements. In *real* or computational 'experiments', in contrast, of course only one of all possible outcomes is measured at a given time, and the measurements have to be repeated on a quite large number of individually prepared systems in order to determine the probabilties.

Show and discuss how such a measurement statistics approaches the quantum-mechanical probabilities for a sufficiently large number of measurements and, especially,

a) for Pauli measurements on 3- and 4-qubit GHZ and W states;

b) for Bell-state measurements on the 3-qubit GHZ and Dicke states;

Please, make a proper distinction of the possible measurements, the qubits to be measured as well as a graphical comparison with regard to the number of measurements and the calculated probabilities.

Task (Local operations on entangled states):

The GHZ and W states are known to belong to (topologically) different classes of entangled states that cannot be transformend into each other by applying just local (single-qubit) operations. Apply one or several single-qubit gates in order to:

a) attempt minimizing the trace distance and the Hilbert-Schmidt distance between the 3-qubit GHZ and W state;

b) do the same but in terms of the (so-called) fidelity.

c) Moreover, is it possible (and, if yes, how) to transform the 3-qubit Dicke states by single-qubit gates into a GHZ state?

Task (Quantum data compression):

Recently, Rozema *et al.* (2014) discussed and realized a protocol in which an ensemble of quantum bits (qubits) can in principle be perfectly suppressed into exponentially fewer qubits. Implement and discuss such a protocol that help compress the information of a 3-qubit (product) state into 2 qubits. Try to extend this scheme towards a larger number of qubits, say, $n = 4, 5, 6, \ldots$

Reference: L. A. Rozema *et al.*, Phys. Rev. Lett. **113** (2014) 160504.

Task (Entanglement witness):

An entanglement *witness* is a functional of the density operator that helps distinguish between the entangled and separable states of a system. Such entanglement witnesses can be either linear or nonlinear functionals of the density matrix. Explore and compare such entanglement witnesses for the two-qubit Bell and diagonal Bell states. Moreover, show how a device-independent entanglement witness (DIEW) can be calculated for two- and three-qubit states. Evaluate and discuss this witness especially for the 3-qubit GHZ, W and Dicke states.

References: J.T. Barreiro *et al.*, Nature Physics **9** (2013) 539; J. Dai *et al.*, Phys. Rev. Lett. **113** (2014) 170402.

Task (Quantum teleportation):

Explain and demonstrate how teleportation can be used to transfer some (unknown) one- and two-qubit quantum state from Alice to Bob. Implement and demonstrate such a teleportation for the Bell and diagonal Bell states. Moreover, how can such a scheme be extended towards (entangled) 3- and multi-qubit systems ?

Reference: M. Fuwa *et al.*, Phys. Rev. Lett. **113** (2014) 223602.

Task (One-way quantum computations):

So-called *one-way* or *measurement-based quantum computing* refers to a method in which an entangled resource state is first prepared, and where then single-qubit measurements are performed with forward control in order to 'steer' a system into a desired state. Often, the resource state is either a cluster or graph state. – Show and explain step-wise how a (given) general single-qubit [SU/(2)] operation can be realized by some subsequent measurements of a five-qubit chain state.

Reference: R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86** (2001) 5188.

Task (Bound hydrogenic wavefunctions with B-splines):

Using basis sets to solve the Schrödinger equation is a popular alternative to finite-difference methods. So-called *B-spline* basis sets are well suited for different reasons. An introduction to B-splines will be given in the last two lectures.

Implement a procedure which solves the radial Schrödinger equation for s-states of the hydrogen atom (or hydrogen-like ions).

a) Compare the energies of the calculated bound states to the exact values. Try different breakpoint sequences on different intervals $[0, r_{\max}]$

b) Calculate the expectation value of r for the hydrogen ground state and compare to the exact value of $\frac{3a_0}{2}$.

A Julia module which evaluates B-splines and their derivatives can be provided by us on request.

Reference: H. Bachau *et al.*, Rep. Prog. Phys. **64** (2001) 1815, sections 1–3.

Task (Continuum wavefunctions with B-splines):

Using basis sets to solve the Schrödinger equation is a popular alternative to finite-difference methods. So-called *B-spline* basis sets are well suited for different reasons. An introduction to B-splines will be given in the last two lectures.

Implement a procedure which solves the radial Schrödinger equation for s-states of the hydrogen atom (or hydrogen-like ions).

a) Solutions with positive energies describe continuum electrons. Try different breakpoint sequences on different intervals $[0, r_{\max}]$. How does the box size $r_{\max}$ affect the solutions?

b) Implement the Galerkin method (cf. Bachau *et al.* 2001, p. 1831) to create continuum waves of arbitrary energy.

A Julia module which evaluates B-splines and their derivatives can be provided by us on request.

Reference: H. Bachau *et al.*, Rep. Prog. Phys. **64** (2001) 1815, sections 1–3.